Uniformly exponentially stable approximation for the transmission line with varying structure parameters

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Abstract

In this note we study an ideal transmission line, which is described by the telegraph's equation with variable coefficients, from the viewpoints of numerical analysis and control theory. Because the semi-discretized systems of original system are not suitable to discusses the uniform exponential stability, we perform a similar transform to the continuous system and obtain an intermediate system, which is conveniently studied. Firstly, we investigate uniform exponential stability for the intermediate system by a so called average central-difference semi-discretization scheme and direct Lyapunov function method. The proof is parallel to that of the continuous case. Secondly, the stability and consistence of numerical approximating scheme are presented by the Trotter-Kato Theorem. At last, a semi-discretization scheme of the original system is proposed by the inverse transform. All results for the intermediate system are then translated into the original system. As the byproduct of the main results, the uniform observability problem is also addressed. Furthermore, the effectiveness of the numerical approximating algorithms is verified by numerical simulations.

Keywords: Transmission line; semi-discretization; finite difference; uniform exponential stability; uniform observability

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1 Introduction

Space semi-discretization, which transforms PDEs into ODEs, is a first natural step in the process of numerical discretization of PDEs. Many researchers addressed this problem from different viewpoints in the past decades. Mathematicians mainly focused on stability and consistency of the approximating algorithm and designed many profound discrete schemes [1]-[3]. Physicists aimed at preserving structural invariants and structural properties of the continuous system. Several recent works tackled this challenge for infinite-dimensional port-Hamiltonian systems. The Dirac structure of the system is preserved by mixed finite element or finite difference methods [4]-[6].

However, when preserving control properties, such as passivity, exponential stability and observability etc, are considered, detailed and elegant analysis is further needed on the basis of stability and consistency of the approximating algorithm [7]-[11]. Banks, Ito and Wang firstly pointed out that the wave equation with boundary damping didn't inherit the exponential stability of the continuous one when the wave equation is discretized by the classical finite difference and finite element schemes in [12]. In the same time, Glowinski, Li and Lions showed that the exact controllability property wasn't preserved for some discretization process in [13]. Lately, Infante and Zuazua discussed the problem of boundary observability of wave equation, i.e., the problem of whether the total energy of solutions can be estimated uniformly in terms of the energy concentrated on the boundary as the net-spacing approaching zero. They derived negative answer due to the existence of high frequency spurious solutions for both finite-difference and finite-element semi-discretizations. A uniform bound was obtained in a subspace of solutions generated by the low frequencies of the discrete system [14]. This method is in fact a filtering strategy which has been widely used in control theory [15]-[16]. Zuazua gave comprehensive survey on the observation and control of waves approximated by finite difference methods in [17].

To overcome the obstacle caused by high frequency spurious modes, several solutions were proposed. One suggestion is to use mixed finite element method [4]-[6] to obtain the uniform controllability of the conserved wave equation [18]. Some other remedies of damping out high frequencies include Tychonoff regularization [13], two-grid algorithms [19] and non-uniform numerical meshes [20]. Introduction of a vanishing viscosity term at whole domain of the spatial variables is another popular approach [21]-[23]. Very recently, Liu and Guo introduced an average operator for the time derivative of classical finite difference for the wave equation with boundary damping and showed that the scheme uniformly preserves the exponential decay of the continuous system [25]. Using the same idea, Xu, Guo and Zheng proposed two completely different finite-difference schemes for uniform exponential approximations of wave equation with local viscosity damping [26]-[28]. Guo and Zheng *et al.* generalized the results of [25]-[28] to the coupled heat-wave system [29] and the Schrödinger equation on $L^2(0,1)$ space [30].

Inspired by above works, we shall study the transmission line with varying capacity and inductance, which is described by the telegraph's equation with variable coefficients, from the viewpoints of numerical analysis and control theory in this note. It is well known that in engineering applications, there are three common physical systems that carry waves: the electrical transmission line, the flexible string, and the compressible fluid [4], [31], [32, Section 7.1]. Since the electrical transmission line is essentially the wave, the effective methods of [25]-[29] can be employed to study it numerically. But the variable coefficients appearing in the space derivatives of PDEs bring some new troubles in construing suitable numerical approximating algorithm to preserve exponential stability. To the best of our knowledge, classical finite-difference method for the spatial semi-discretization of the telegraph's equation of this note hardly preserves the passivity, not even the exponential stability of the continuous system.

To overcome these difficulties we first perform a similar transform to the original system and obtain an intermediate system, which is easily studied. A semi-discretization scheme, which is called average central-difference (see Section 4), is proposed for the intermediate system. We then investigate the uniform exponential stability of the discrete systems utilizing the method paralleling to that of the continuous case. We also reveal how the mechanism of boundary feedback control uniformly exponentially stabilizes the discrete system (see Remark 2). Secondly, the stability and consistence of numerical approximating algorithm are presented by the Trotter-Kato Theorem. At last, a discretization scheme of the original system is proposed by another similar transform. All results on the intermediate system are then translated into the original system. As the byproduct of the main results, the uniform observability is also presented. Furthermore, to show the effectiveness of the numerical approximating algorithms, we perform several numerical simulations.

Thus, the contribution of this work is four-fold:

- Give a new method to study numerical solution of PDEs with variable coefficients. In the meanwhile, explain how the feedback control uniformly exponentially stabilize the discrete system.
- Extend the results of uniform exponential stability of [21] and [25]-[28], and uniform observability of [14] and [17] from simple models to more complex system.
- Provide demonstration to study uniform exponential stability of other complex models described by PDEs such as wave-wave coupled equations, beam equations and so on.
- The results of uniform exponential stability and uniform observability have potential applications in uniform controllability, the approximation of the control problem and state reconstruction etc.

The structure of this paper is as follows. In section 2, the transmission line system is introduced and some results about exponential stability and exact observability of the intermediate system are presented. In section 3, the

uniform exponential stability and uniform observability of discrete systems are obtained. In section 4, the stability and consistence of numerical approximating algorithm is derived by the Trotter-Kato Theorem. In section 5, a discretization scheme of the original system is proposed by similar transform. The uniform exponential stability of the original system is proved and convergence analysis of the discretization scheme is made based on the result of section 4. In section 6, numerical simulations are performed. In section 7, we give some concluding remarks.

2 Results of continuous model

In this section, we introduce the model discussed in this paper and present some known results about the exponential stability of continuous systems. Consider the transmission line on the spatial interval $[0, 1]$:

$$\begin{cases} \frac{\partial}{\partial t}Q(t,x) = -\frac{\partial}{\partial x}\left(\frac{\phi(t,x)}{L(x)}\right), \ x \in (0,1), \\ \frac{\partial}{\partial t}\phi(t,x) = -\frac{\partial}{\partial x}\left(\frac{Q(t,x)}{C(x)}\right), \ t > 0, \\ V(t,0) = 0, \quad V(t,1) = RI(t,1), \ R > 0 \\ Q(0,x) = Q^0(x), \quad \phi(0,x) = \phi^0(x). \end{cases} \tag{1}$$

Here $Q(t, x)$ is the charge at position $x \in [0, 1]$ and time $t > 0$, and $\phi(t, x)$ is the magnetic flux at position x and time t. $C(x)$ is the distributed capacity and $L(x)$ is the distributed inductance. The voltage and the current are given by $V = Q/C$ and $I = \phi/L$, respectively. $V(t, 1) = \beta I(t, 1)$ is the boundary feedback and R is the feedback gain constant. $(Q^0(\cdot), \phi^0(\cdot)) \in [L^2(0, 1)]^2$ are initial configuration of the transmission line model. The energy of this system is given by

$$E(t) = \frac{1}{2}\int_0^1 \frac{|\phi(t,x)|^2}{L(x)} + \frac{|Q(t,x)|^2}{C(x)}\mathrm{d}x. \tag{2}$$

It follows from the exercises 7.1 and 9.1 of [32] that the system (1) is exponentially stable with respect to the energy $E(t)$. In order to study the uniform exponential stability in a convenient manner, we begin from the following system on V and I:

$$\begin{cases} C(x)V_t(t,x) = -I_x(t,x), \ x \in (0,1), \\ L(x)I_t(t,x) = -V_x(t,x), \ t > 0, \\ V(t,0) = 0, \quad V(t,1) = RI(t,1), \\ V(0,x) = V^0(x), \quad I(0,x) = I^0(x). \end{cases} \tag{3}$$

The energy of system (3) is also given by

$$E(t) = \frac{1}{2}\int_0^1 C(x)|V(t,x)|^2 + L(x)|I(t,x)|^2\mathrm{d}x \tag{4}$$

and has different expression in light of $V = Q/C$ and $I = \phi/L$. To give some clues in discrete case, we apply the method of Lyapunov function to obtain the exponential stability of the continuous system (3). For this purpose, we assume that the capacity function $C(x)$ and the inductance function $L(x)$ satisfy:

H_1: $C(x) > 0$, $L(x) > 0$, $\forall x \in [0, 1]$;

H_2: $C(x)$, $L(x) \in C^1[0, 1]$ and $C(x) \leq K$, $L(x) \leq K$ for some positive constant K;

H_3: $C'(x) > 0$, $L'(x) > 0$, $\forall x \in [0, 1]$.

It should point out that H_3 is not applied in the proof of the main result, see for instance Theorem 3. It is only used in the method of Lyapunov function to verify the exponential stability of the continuous system. If one applies the method of [32, Lemma 9.1.3], H_3 is also useless.

Theorem 1 *Under the conditions H_1-H_3, for any $V^0(\cdot)$, $I^0(\cdot) \in L^2[0, 1]$, there exist two constants M and ω such that the energy of the solution to the system (3) satisfies*

$$E(t) \leq Me^{-\omega t}E(0). \tag{5}$$

Proof: It is easy to see that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) &= -\int_0^1 V(t,x)I_x(t,x) + V_x(t,x)I(t,x)\mathrm{d}x \\ &= -[V(t,x)I(t,x)]\,\big|_0^1 = -R|I(t,1)|^2.\end{aligned} \tag{6}$$

Introduce the auxiliary function

$$\rho(t) = -\int_0^1 xC(x)L(x)V(t,x)I(t,x)\mathrm{d}x$$

and Lyapunov function $F(t) = E(t) + \varepsilon\rho(t)$. Here $\varepsilon \in (0, 1/K)$ is a parameter. On one hand, we have $|\rho(t)| \leq KE(t)$ by using condition H_2 and Cauchy inequality and $F(t)$ is equivalent to $E(t)$:

$$(1 - K\varepsilon)E(t) \leq F(t) \leq (1 + K\varepsilon)E(t). \tag{7}$$

The parameter $0 < \varepsilon < 1/K$ ensures that $F(t)$ is positive definite. On the other hand, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \\ &= [L(1) + C(1)R^2]|I(t,1)|^2 - 2E(t) - \int_0^1 xC'(x)|V(t,x)|^2 + \\ &\quad xL'(x)|I(t,x)|^2\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\rho(t),\end{aligned} \tag{8}$$

It follows from the above equality and assumption H_3 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \leq \frac{L(1)+C(1)R^2}{2}|(t,1)|^2 - E(t). \tag{9}$$

By differentiating $F(t)$ and using (6), (7) and (9), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F(t) &= \frac{\mathrm{d}}{\mathrm{d}t}E(t) + \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)\\ &\leq -\left[R - \frac{\varepsilon(L(1)+C(1)R^2)}{2}\right]|I(t,1)|^2 - \frac{\varepsilon}{1+K\varepsilon}F(t)\end{aligned}$$

Finally, choosing ε to ensure $R - \left(\varepsilon(L(1)+C(1)R^2)\right)/2 \geq 0$ and applying the comparison principle (see Section 3.1 of [33]) and (7), we get

$$E(t) \leq Me^{-\omega t}E(0)$$

with $M = (1+K\varepsilon)/(1-K\varepsilon)$ and $\omega = \varepsilon/(1-K\varepsilon)$. Therefore we obtain the exponential stability of the system (3) and complete the proof of the theorem. $\square$

Applying the auxiliary function $\rho(t)$, one can also prove an observability inequality.

Theorem 2 *Under the assumptions H_1-H_3, for any $V^0(\cdot), I^0(\cdot) \in L^2[0,1]$, there exist time $\tau > 2K$ and a constant C_τ such that the solution to the system*

$$\begin{cases} C(x)V_t(t,x) = -I_x(t,x), \ x \in (0,1),\\ L(x)I_t(t,x) = -V_x(t,x), \ t > 0,\\ V(t,0) = 0, \quad I(t,1) = 0,\\ V(0,x) = V^0(x), \quad I(0,x) = I^0(x). \end{cases} \tag{10}$$

satisfies the observability inequality

$$\int_0^\tau |V(t,1)|^2 dt \geq C_\tau E(0). \tag{11}$$

Proof: It follows from (6) that the energy of the system (10) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -[V(t,x)I(t,x)]|_0^1 = 0. \tag{12}$$

Similarly, from (8) and (10) we have

$$\begin{aligned}&2\frac{\mathrm{d}}{\mathrm{d}t}\rho(t)\\ &= C(1)|V(t,1)|^2 - 2E(t) - \int_0^1 xC'(x)|V(t,x)|^2 + xL'(x)|I(t,x)|^2\mathrm{d}x,\end{aligned}$$

which implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \leq \frac{C(1)}{2}|V(t,1)|^2 - E(t)$$

with the help of assumption H_3. Integrating the above inequality from 0 to τ and using (12) and $|\rho(t)| \leq KE(t)$, we get the observability inequality (11) with the constant $C_\tau = 2(\tau - 2K)/C(1)$. Therefore we complete the proof of the theorem. $\square$

For convenience in the convergence analysis, we rewrite the exponential stability of the system (3) in the language of semigroup of bounded linear operators.

Remark 1 *Let* $H = \left[L^2(0,1)\right]^2$ *be the state space and define the inner product on* H *by:* $\langle m, \widetilde{m}\rangle_H = \int_0^1 C(x)V(x)\widetilde{V}(x) + L(x)I(x)\widetilde{I}(x)dx$, $\forall m = (V,I), \widetilde{m} = (\widetilde{V}, \widetilde{I}) \in H$. *Define the system operator* A *on* H *through:*

$$A\begin{pmatrix} p(x) \\ q(x)\end{pmatrix} = \begin{pmatrix} -\frac{1}{C(x)}\frac{d}{dx}q(x) \\ -\frac{1}{L(x)}\frac{d}{dx}p(x)\end{pmatrix}, \tag{13}$$

$$D(A) = \left\{\begin{pmatrix} p(\cdot) \\ q(\cdot)\end{pmatrix} \in [H^1(0,1)]^2 : p(0) = 0,\ p(1) = Rq(1)\right\}. \tag{14}$$

Thus, the system (3) can be transformed into abstract Cauchy problem

$$\frac{d}{dt}\begin{pmatrix} V(t,x) \\ I(t,x)\end{pmatrix} = A\begin{pmatrix} V(t,x) \\ I(t,x)\end{pmatrix}, \qquad \begin{pmatrix} V(0,x) \\ I(0,x)\end{pmatrix} = \begin{pmatrix} V^0(x) \\ I^0(x)\end{pmatrix} \in H.$$

A routine method as given in Theorem 3.1.11 of [34] can be applied to show that the operator A *generates a contractive semigroup* $T(t)$ *on* H. *This means that the system (3) has an unique solution for any* $V^0(\cdot), I^0(\cdot) \in L^2(0,1)$. *Theorem 1 further implies that the semigroup* $T(t)$ *is exponentially stable under the assumptions* H_1-H_3, *i.e.*

$$\|T(t)\|_H \leq Me^{-\omega t},$$

here $\|\cdot\|_H$ *is the norm induced by the inner on* H.

3 Uniform exponential stability and uniform observability

Firstly, we introduce the semi-discretization scheme for the system (3). For this purpose, let $N \in \mathbb{N}$ be a positive integer and $h = 1/(N+1)$ mesh size. Insert $N+2$ points and $N+1$ points, denoted by $y_i = ih$ $(i = 0, 1, \cdots, N+1)$ and $x_j = (j+1/2)h$ $(j = 0, 1, \cdots, N)$ respectively, in the domain $[0,1]$. If let f_i be the value of any continuous function $f(x)$ at the node $y_i = ih$ $(i = 0, 1, \cdots, N+1)$, then the notations

$$\delta_x f_j = \frac{f_{j+1} - f_j}{h}, \quad \delta_{\frac{1}{2}} f_j = \frac{f_{j+1} + f_j}{2}$$

denote the central difference operator of $f_x(x)$ and the average operator of $f(x)$ at the node x_j, respectively. Inspired by the works of [21]-[26], we propose the following semi-discretization scheme for (3)

$$\begin{cases} C_j \frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}} V_j(t) = -\delta_x I_j(t), \ L_j \frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}} I_j(t) = -\delta_x V_j(t), \ j = 0, 1, \cdots N, \\ V_0(t) = 0, \quad V_{N+1}(t) = RI_{N+1}(t), \\ V(0, y_i) \approx V^0(y_i) = V_i^0, \ I(0, y_i) \approx I^0(y_i) = I_i^0, \ i = 0, 1, \cdots N+1. \end{cases} \tag{15}$$

We call this semi-discretization scheme as average central-difference method since the average operator and central-difference operator are applied for the temporal derivative and spatial derivative, respectively.

The energy of discrete system (15) is

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N}\left[C_j\left|\delta_{\frac{1}{2}} V_j(t)\right|^2 + L_j\left|\delta_{\frac{1}{2}} I_j(t)\right|^2\right],$$

which is discrete counterpart of the energy $E(t)$.

Definition 1 *If there exist two constants M and ω independent of t and h such that*

$$E_h(t) \le M e^{-\omega t} E_h(0),$$

then we call that the system (15) is uniform exponentially stable.

The state space of the discrete system (15) is $H_N = \left[\mathbb{R}^{N+1}\right]^2$ with the inner product

$$\langle (p_h, q_h), (u_h, v_h)\rangle_N = h\sum_{j=0}^{N}\left[C_j \delta_{\frac{1}{2}} p_j \delta_{\frac{1}{2}} u_j + L_j \delta_{\frac{1}{2}} q_j \delta_{\frac{1}{2}} v_j\right],$$

for all (p_h, q_h), $(u_h, v_h) \in H_N$. Here $p_h = (p_1, \cdots, p_{N+1})$, $q_h = (q_0, \cdots, q_N) \in \mathbb{R}^{N+1}$, (u_h, v_h) and (p_h, q_h) are the same type of vectors. Moreover, set artificially $k = R^{-1}$, $p_0 = u_0 = 0$, $q_{N+1} = kp_{N+1}$, and $v_{N+1} = ku_{N+1}$, to unify the notations of $\delta_{\frac{1}{2}} p_j$ and $\delta_x u_j$ etc. When $R = 0$, i.e. $q_{N+1} = 0$, the space H_N is still meaningful and denoted by H_N^0 in this case.

To verify the uniform exponential stability of the systems (15), we will follow every step of the proof of Theorem 1. However, we need the following lemma additionally.

Lemma 1 *Let $\{u_i\}_{i=0}^{N+1}$, $\{v_i\}_{i=0}^{N+1}$ and $\{w_i\}_{i=0}^{N+1}$ be sequences consisting of real numbers, then we have*

$$\frac{1}{4}\sum_{i=0}^{N}(u_{i+1} - u_i)(v_{i+1} + v_i)(w_{i+1} + w_i)$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}-u_i)(v_{i+1}-v_i)(w_{i+1}-w_i)\\
&+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}-v_i)(w_{i+1}+w_i)\\
&+\frac{1}{4}\sum_{i=0}^{N}(u_{i+1}+u_i)(v_{i+1}+v_i)(w_{i+1}-w_i)\\
&=u_{N+1}v_{N+1}w_{N+1}-u_0v_0w_0
\end{aligned}
$$

Proof: Extracting the factors from the first two terms and the remaining terms respectively and by simple algebra operations, we obtain

$$
\text{Left}=\frac{1}{2}\sum_{i=0}^{N}[(u_{i+1}-u_i)(v_{i+1}w_{i+1}+w_iv_i)+(u_{i+1}+u_i)(v_{i+1}w_{i+1}-w_iv_i)].
$$

Breaking the brackets in right hand side of the identity above and eliminating the cross terms, one has

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=0}^{N}[(u_{i+1}-u_i)(v_{i+1}w_{i+1}+w_iv_i)+(u_{i+1}+u_i)(v_{i+1}w_{i+1}-w_iv_i)]\\
=&\sum_{i=0}^{N}(u_{i+1}v_{i+1}w_{i+1}-u_iv_iw_i)=u_{N+1}v_{N+1}w_{N+1}-u_0v_0w_0.
\end{aligned}
$$

We complete the proof of the lemma by the identities above. □

Theorem 3 *Under the assumptions H_1 and H_2, the semi-discretized system (15) is uniform exponentially stable.*

Proof: Firstly, differentiating the energy $E_h(t)$ with respect to time t along the solution to (15), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E_h(t)\\
&=-h\sum_{j=0}^{N}\left[\delta_{\frac{1}{2}}V_j(t)\delta_xI_j(t)+\delta_{\frac{1}{2}}I_j(t)\delta_xV_j(t)\right]\\
&=-\frac{1}{2}\sum_{j=0}^{N}(V_{j+1}(t)-V_j(t))(I_{j+1}(t)+I_j(t))\\
&\quad+\frac{1}{2}\sum_{j=0}^{N}(I_{j+1}(t)-I_j(t))(V_{j+1}(t)+V_j(t))
\end{aligned}
$$

$$= -\frac{1}{2}\sum_{j=0}^{N}(V_{j+1}(t)I_{j+1}(t) - V_j(t)I_j(t))$$
$$= -[V_{N+1}(t)I_{N+1}(t) - V_0(t)I_0(t)]. \quad (16)$$

This means that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_h(t) = -R|I_{N+1}(t)|^2. \quad (17)$$

Secondly, assume that ε is a parameter and define Lyapunov functions by $F_h(t) = E_h(t) + \varepsilon\rho_h(t)$. Here $\rho_h(t)$ are auxiliary functions given by $\rho_h(t) = -h\sum_{j=0}^{N} C_j L_j \delta_{\frac{1}{2}} y_j \delta_{\frac{1}{2}} V_j(t)\delta_{\frac{1}{2}} I_j(t)$. On one hand, it is easy to see that $|\rho_h(t)| \leq KE_h(t)$. This implies that Lyapunov functions $L_h(t)$ are equivalent to the discrete energy $E_h(t)$, i.e.

$$(1 - \varepsilon K)E_h(t) \leq F_h(t) \leq (1 + \varepsilon K)E_h(t). \quad (18)$$

The parameter $0 < \varepsilon < 1/K$ ensures that Lyapunov functions $F_h(t)$ are positive definite.

On the other hand, differentiating the auxiliary function $\rho_h(t)$ and applying (15), we derive

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_h(t) = h\sum_{j=0}^{N} C_j\delta_{\frac{1}{2}} y_j\delta_{\frac{1}{2}} I_j(t)\delta_x I_j(t) + h\sum_{j=0}^{N} L_j\delta_{\frac{1}{2}} y_j\delta_{\frac{1}{2}} V_j(t)\delta_x V_j(t)$$
$$= \frac{1}{4}\sum_{j=0}^{N} L_j(y_{j+1} + y_j)(I_{j+1}(t) - I_j(t))(I_{j+1}(t) + I_j(t))$$
$$+\frac{1}{4}\sum_{j=0}^{N} C_j(y_{j+1} + y_j)(V_{j+1}(t) - V_j(t))(V_{j+1}(t) + V_j(t)). \quad (19)$$

Using $y_{N+1} = 1$, $y_0 = 0$, $y_{j+1} - y_j = h$ and Lemma 1, we obtain

$$\frac{1}{4}\sum_{j=0}^{N} L_j(y_{j+1} + y_j)(I_{j+1}(t) - I_j(t))(I_{j+1}(t) + I_j(t))$$
$$= -\frac{h}{8}\sum_{j=0}^{N} L_j|I_{j+1}(t) - I_j(t)|^2 - \frac{h}{2}\sum_{j=0}^{N} L_j|\delta_{\frac{1}{2}} I_j(t)|^2 + L_N\frac{|I_{N+1}(t)|^2}{2} \quad (20)$$

and

$$\frac{1}{4}\sum_{j=0}^{N} C_j(y_{j+1} + y_j)(V_{j+1}(t) - V_j(t))(V_{j+1}(t) + V_j(t))$$
$$= -\frac{h}{8}\sum_{j=0}^{N} C_j\left|V_{j+1}(t) - V_j(t)\right|^2 - \frac{h}{2}\sum_{j=0}^{N} C_j\left|\delta_{\frac{1}{2}} V_j(t)\right|^2 + C_N\frac{|V_{N+1}(t)|^2}{2} \quad (21)$$

Combining (19) and (20)-(21), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_h(t) \leq \alpha |I_{N+1}(t)|^2 - E_h(t), \tag{22}$$

in which $\alpha = 1/2\left(L_N + C_N R^2\right)$. Note that the inequality (22) is a perfect counterpart of inequality (8).

Finally, differentiating Lyapunov function $F_h(t)$ and using (17)-(18) and (22), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F_h(t) = \frac{\mathrm{d}}{\mathrm{d}t}E_h(t) + \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\rho_h(t) \leq -\left(R - \varepsilon\alpha\right)\mathsf{I}(\mathsf{t},1)^2 - \frac{\varepsilon}{1+K\varepsilon}F_h(t).$$

Choosing ε to ensure $R - \varepsilon\alpha \geq 0$ and applying the comparison principle and (18), we get

$$E_h(t) \leq Me^{-\omega t}E_h(0).$$

M and ω are the same as in the proof of Theorem 1. Therefore we obtain uniform exponential stability of the system (15) and complete the proof of the theorem. □

Similarly, using the auxiliary function $\rho(t)$, we can also prove the discrete versions of observability inequality 11 hold uniformly with respect to step h. This is the main content of the following theorem.

Theorem 4 *Propose the semi-discrentized scheme of the system (10) as follows*

$$\begin{cases} C_j\frac{d}{dt}\delta_{\frac{1}{2}}V_j(t) = -\delta_x I_j(t),\ L_j\frac{d}{dt}\delta_{\frac{1}{2}}I_j(t) = -\delta_x V_j(t),\ j = 0,1,\cdots N, \\ V_0(t) = 0, \quad I_{N+1}(t) = 0, \\ V(0,y_i) \approx V^0(y_i) = V_i^0,\ I(0,y_i) \approx I^0(y_i) = I_i^0,\ i = 0,1,\cdots N+1. \end{cases} \tag{23}$$

Then under the assumptions H_1-H_2, for any $(V_1^0,\cdots,V_{N+1}^0,I_0^0,I_1^0,\cdots,I_N^0)^\top \in H_N^0$ there exist time $\tau > 2K$ and a constant C_τ independent of h such that the solution to the system (23) satisfies the uniform observability inequality

$$\int_0^\tau |V_{N+1}(t)|^2 dt \geq C_\tau E_h(0). \tag{24}$$

Proof: It follows from (23) and (17) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E_h(t) = -R\,|I_{N+1}(t)]| = 0. \tag{25}$$

Similarly, from (23) and (19)-(21) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \leq \frac{C(1)}{2}|V_{N+1}(t)|^2 - E_h(t).$$

Integrating the above inequality from 0 to τ and using (25) and $|\rho_h(t)| \leq KE_h(t)$, we get the uniform observability inequalities (24) with the constant $C_\tau = 2(\tau - 2K)/C_{N+1}$. Therefore we complete the proof of the theorem. $\square$

At the end of this section, we explain the mechanism of boundary feedback control of our numerical scheme how to uniformly exponentially stabilizing the discrete system (15).

Remark 2 *Let $V_h(t) = (V_1(t), \cdots, V_{N+1}(t))^\top$ and $I_h(t) = (I_0(t), \cdots, I_N(t))^\top$ be the unknown variables of (15). We solve $V_1'(t)$ from (15) by letting $j = 0$ in the first equation of (15) since $V_0(t) = 0$. Using $V_1'(t)$, we can solve $V_2'(t)$ by letting $j = 1$ in the first equation of (15). Repeating this process, we can separate all components of $V_h'(t)$. But we obtain $I_h'(t)$ by the converse process in view of $I_{N+1}(t) = kV_{N+1}(t)$. That is the last component $I_N'(t)$ is the starting point of solving $I_h'(t)$ from the second equation of (15). Thus we obtain the equivalent form of the discrete system (15)*

$$\frac{d}{dt}(V_h(t), I_h(t))^\top = (B_1 I_h(t), \cdots, \ B_{N+1} I_h(t), \ D_0 V_h(t), \cdots, \ D_N V_h(t))^\top, \quad (26)$$

in which

$$\begin{aligned}
B_1 I_h(t) &= -2\frac{\delta_x I_0(t)}{C_0}, \\
B_{j+1} I_h(t) &= -2\frac{\delta_x I_j(t)}{C_j} - B_j I_h(t), \ j = 1, \cdots, N-1, \\
B_{N+1} I_h(t) &= -2\frac{kV_{N+1}(t) - I_N(t)}{hC_N} - B_N I_h(t), \\
D_N V_h(t) &= -2\frac{\delta_x V_N(t)}{L_N} - kB_{N+1} I_h(t), \\
D_i V_h(t) &= -2\frac{\delta_x V_i(t)}{L_i} - D_{i+1} V_h(t), \ i = 0, 1, \cdots, N-1
\end{aligned}$$

satisfy the recursion relations and are all mappings from $\mathbb{R}^{N+1}$ to $\mathbb{R}$.

If you track the feedback control $kV_{N+1}(t)$, which corresponds to $kB_{N+1}I_h(t)$, in the dynamical system (26), you can find out that it firstly enters into the channel $I_N'(t)$ and then every channel of the system (26) one by one. This is caused by the average operator for the time derivative. Without average operator, the semi-discretization scheme (15) degenerates into the classical central difference scheme and the feedback control only appears one channel. Furthermore, it has pointed out that there is no uniform exponential stability for this approximating scheme in [21]. More information is given in section 6. This is the main mechanism of boundary feedback control uniformly exponentially stabilizes the discrete system. This has the same effect with the mixed finite element method in [18].

4 Convergence analysis

We will show that the solution to the system (15) converges to the corresponding solution of the system (2) in the sense of Trotter-Kato. Since the convergence analysis for (3.8) and (10) is the special case of $R = 0$, we skip it.

For every $n = 1, 2, \cdots$, there exist bounded linear operators $P_n : X \to X_n$ and $E_n : X_n \to X$ satisfying

(A_1) There exist two positive constants M_1 and M_2 such that $\|E_n\| \leq M_1$ and $\|P_n\| \leq M_2$,

(A_2) $\|E_n P_n x - x\| \to 0$ as $n \to \infty$ for all $x \in X$,

(A_3) $P_n E_n = I_n$, where I_n is the identity operator on X_n.

The notation $B \in G(M, \omega, X)$ with $M > 1$ and $\omega \in \mathbb{R}$, means that B is the infinitesimal generator of a C_0-semigroup $S(t)$, $t > 0$, satisfying $\|S(t)\| \leq Me^{\omega t}$. The Trotter-Kato Theorem for approximating a linear C_0-semigroup $S(t)$ on a Banach space X is as follows.

Theorem 5 *(**Trotter-Kato** [35]). Assume that $(A1)$ and $(A3)$ are satisfied. Let B resp. B_n be in $G(M, \omega, X)$ resp. in $G(M, \omega, X_n)$ and let $S(t)$ and $S_n(t)$ be the semigroups generated by B and B_n on Banach spaces X and X_n, respectively. Then the following statements are equivalent*

(a) There exists a $\lambda_0 \in \rho(B) \cap \bigcap_{n=1}^{\infty} \rho(B_n)$ such that, for all $x \in X$,

$$\|E_n(\lambda_0 - B_n)^{-1} P_n x - (\lambda_0 - B)^{-1} x\| \to 0, \quad \textit{as } n \to \infty. \tag{27}$$

(b) For every $x \in X$ and $t \geq 0$,

$$\|E_n S_n(t) P_n x - S(t)x\| \to 0, \quad \textit{as } n \to \infty \tag{28}$$

uniformly on bounded t-intervals.

Note that the assumptions $B_n \in G(M, \omega, X_n)$, or equivalently $\|S_n(t)\|_n \leq Me^{\omega t}$, $n = 1, 2, \cdots$, usually is called the stability property of the approximations, whereas statement (a) is called the consistency property of the approximations. However, one may face some major difficulties when one applies Theorem 5 to perform convergence analysis. The most difficult one is how to verify the consistency property (a). The following property, which can replace (a) by a condition involving convergence of the operators B_n to B in some sense, is useful in this part [35].

Proposition 1 *Assume that the assumptions of Theorem 5 are satisfied. Then statement (a) of Theorem 5 is equivalent to $(A2)$ and the following two statements:*

$(C1)$ There exists a subset $D \subseteq D(B)$ such that $\overline{D} = X$ and $(\lambda_0 I - B)^{-1} D = X$ for a $\lambda_0 > \omega$.

$(C2)$ For all $u \in D$ there exists a sequence $(\overline{u}_n)_{n \in \mathbb{N}}$ with $\overline{u}_n \in D(B_n)$ such that

$$\lim_{n \to \infty} E_n \overline{u} = u, \qquad \lim_{n \to \infty} E_n B_n \overline{u} = Bu. \tag{29}$$

Now we give the convergence analysis of our systems. In light (26), the approximating operators A_N are obviously defined by, for any $(p_h q_h) \in H_N$,

$$A_N \begin{pmatrix} p_h \\ q_h \end{pmatrix} = \begin{pmatrix} Bq_h \\ Dp_h \end{pmatrix}, \text{ with } Bq_h = \begin{pmatrix} B_1 q_h \\ \vdots \\ B_{N+1} q_h \end{pmatrix}, \; Dp_h = \begin{pmatrix} D_0 p_h \\ \vdots \\ D_N p_h \end{pmatrix}. \tag{30}$$

By the same operations as in (16), one has

$$\left\langle A_N \begin{pmatrix} p_h \\ q_h \end{pmatrix}, \begin{pmatrix} p_h \\ q_h \end{pmatrix} \right\rangle_N = -h \sum_{j=0}^{N} \left[\delta_{\frac{1}{2}} p_j \delta_x q_j + \delta_x p_j \delta_{\frac{1}{2}} q_j \right] = -k|p_{N+1}|^2 \leq 0.$$

This means that $A_N \in G(1, 0, H_N)$. However it follows from of Remark 1 that A generates some contractive semigroup, i.e. $A \in G(1, 0, H)$. This shows that the discrete scheme of (15) is stable.

Let χ_S be the characteristic function of the set S and define the extension operators $E_N : H_N \to H$:

$$E_N \begin{pmatrix} p_h \\ q_h \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{N} (\delta_{\frac{1}{2}} p_i) \chi_{(y_i, y_{i+1}]} \\ \sum_{i=0}^{N} (\delta_{\frac{1}{2}} q_i) \chi_{(y_i, y_{i+1}]} \end{pmatrix},$$

Choose the dense subset $D \triangleq D(A) \bigcap (C^2[0,1])^2$ of H. For any $(u(\cdot), v(\cdot))^\top \in D$, set $\overline{u} = (u(y_1), \cdots, u(y_{N+1}))^\top$, $\overline{v} = (v(y_0), \cdots, v(y_N))^\top$ and $\overline{U} = (\overline{u}, \overline{v})^\top$. By $u(y_0) = u(0) = 0$ and $v(1) = v(y_{N+1}) = ku(y_{N+1}) = ku(1)$, it is easy to see

$$E_N \overline{U} = E_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{N} \left(\frac{u(y_{i+1}) + u(y_i)}{2} \right) \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^{N} \left(\frac{v(y_{i+1}) + v(y_i)}{2} \right) \chi_{[y_i, y_{i+1}]} \end{pmatrix},$$

and

$$E_N A_N \overline{U} = E_N \begin{pmatrix} B\overline{v} \\ C\overline{u} \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{N} \left(\frac{v(y_{i+1}) - v(y_i)}{hC_i} \right) \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^{N} \left(\frac{u(y_{i+1}) - u(y_i)}{hL_i} \right) \chi_{[y_i, y_{i+1}]} \end{pmatrix}.$$

Furthermore, we have:

$$\begin{aligned} & E_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \\ = & \begin{pmatrix} \sum_{i=0}^{N} \left[\frac{u(y_{i+1}) + u(y_i)}{2} \right] \chi_{[y_i, y_{i+1}]} - \sum_{i=0}^{N} u(x) \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^{N} \left[\frac{v(y_{i+1}) + v(y_i)}{2} \right] \chi_{[y_i, y_{i+1}]} - \sum_{i=0}^{N} v(x) \chi_{[y_i, y_{i+1}]} \end{pmatrix} \\ = & \frac{1}{2} \begin{pmatrix} \sum_{i=0}^{N} [u(y_{i+1}) - u(x) + u(y_i) - u(x)] \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^{N} [v(y_{i+1}) - v(x) + v(y_i) - v(x)] \chi_{[y_i, y_{i+1}]} \end{pmatrix} \\ = & \frac{1}{2} \begin{pmatrix} \sum_{i=0}^{N} [u'(\xi_{i+1}^u)(y_{i+1} - x) + u'(\xi_i^u)(x - y_i)] \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^{N} [v'(\xi_{i+1}^v)(y_{i+1} - x) + v'(\xi_i^v)(x - y_i)] \chi_{[y_i, y_{i+1}]} \end{pmatrix}, \end{aligned}$$

in which the mean value Theorem is applied and $u(x),\ v(x) \in C^1[0,1]$ implies that $|u'(\xi_i^u)|$, $|u'(\xi_{i+1}^u)|$, $|v'(\xi_i^v)|$ and $|v'(\xi_{i+1}^v)|$ are uniformly bounded with

respect to $i = 0, 1, \cdots, N$. Let Γ be their common upper bound and so we have

$$\left\| E_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_N^2 \leq \frac{\Gamma^2 h^2}{2} \left\| \begin{pmatrix} \sum_{i=0}^N \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^N \chi_{[y_i, y_{i+1}]} \end{pmatrix} \right\|_N^2 \to 0, \quad \text{as } N \to \infty.$$

Similarly, for $E_N A_N - A$ we obtain

$$\begin{aligned} & E_N A_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - A \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \\ = & \begin{pmatrix} \sum_{i=0}^N \left[\frac{v(y_{i+1}) - v(y_i)}{hC_i} - \frac{v'(x)}{C(x)} \right] \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^N \left[\frac{u(y_{i+1}) - u(y_i)}{hL_i} - \frac{u'(x)}{L(x)} \right] \chi_{[y_i, y_{i+1}]} \end{pmatrix} \\ = & \begin{pmatrix} \sum_{i=0}^N \left[\frac{v'(\eta_{i+1}^v)}{C_i} - \frac{v'(x)}{C(x)} \right] \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^N \left[\frac{u'(\eta_{i+1}^u)}{L_i} - \frac{u'(x)}{L(x)} \right] \chi_{[y_i, y_{i+1}]} \end{pmatrix} \\ = & \begin{pmatrix} \sum_{i=0}^N \frac{C(x)[v'(\eta_{i+1}^u) - v(x)] + [C(x) - C_i] v'(x)}{C_i C(x)} \chi_{[y_i, y_{i+1}]} \\ \sum_{i=0}^N \frac{L(x)[u'(\eta_{i+1}^u) - u(x)] + [L(x) - L_i] u'(x)}{L_i L(x)} \chi_{[y_i, y_{i+1}]} \end{pmatrix}. \end{aligned}$$

By the same idea as above, we can show that $E_N A_N \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} - A \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$ converges to zero as $N \to \infty$ since $C(x)$, $L(x) \in C^1[0,1]$ and $u(x)$, $v(x) \in C^2[0,1]$. Therefore, the statement (b) in Proposition 1 holds.

Finally, construct the projecting operators $P_N : H \to H_N$ in light of the expressions of the extensions E_N by

$$P_N \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} I^1 u(x) \\ I^2 v(x) \end{pmatrix},$$

with

$$I^1 u(x) = 2 \begin{pmatrix} I_0^1 u(x) \\ \vdots \\ I_N^1 u(x) \end{pmatrix}, \quad I^2 v(x) = 2 \begin{pmatrix} I_0^2 v(x) \\ \vdots \\ I_N^2 v(x) \end{pmatrix},$$

and

$$\begin{aligned} I_0^1 u(x) &= h^{-1} \int_{y_0}^{y_1} u(x) \mathrm{d}x, \\ I_i^1 u(x) &= h^{-1} \int_{y_i}^{y_{i+1}} u(x) \mathrm{d}x - I_{i-1}^1 u(x), \ i = 1, 2, \cdots, N \\ I_N^2 v(x) &= h^{-1} \int_{y_N}^{y_{N+1}} u(x) \mathrm{d}x - k I_N^1 v(x), \end{aligned}$$

$$I_j^2u(t)=h^{-1}\int_{y_j}^{y_{j+1}}v(x)\mathrm{d}x-I_{j+1}u(t),\ j=0,\ 1,\ \cdots,\ N-1.$$

It is easy to show that E_N and P_N satisfy $(A1)$ and $(A3)$.

To prove that E_N and P_N satisfy $(A2)$, we firstly assume $(u(x),v(x))\in D(A)$. With this assumption we have

$$E_NP_N\begin{pmatrix}u(x)\\v(x)\end{pmatrix}=h^{-1}\begin{pmatrix}\sum_{i=0}^N\int_{y_i}^{y_{i+1}}u(x)\mathrm{d}x\chi_{[y_i,y_{i+1}]}\\\sum_{i=0}^N\int_{y_i}^{y_{i+1}}v(x)\mathrm{d}x\chi_{[y_i,y_{i+1}]}\end{pmatrix}=\begin{pmatrix}\sum_{i=0}^N u(\theta_i^u)\chi_{[y_i,y_{i+1}]}\\\sum_{i=0}^N v(\theta_i^v)\chi_{[y_i,y_{i+1}]}\end{pmatrix},$$

and

$$E_NP_N\begin{pmatrix}u(x)\\v(x)\end{pmatrix}-\begin{pmatrix}u(x)\\v(x)\end{pmatrix}=\begin{pmatrix}\sum_{i=0}^N[u(\theta_i^u)-u(x)]\chi_{[y_i,y_{i+1}]}\\\sum_{i=0}^N[v(\theta_i^v)-v(x)]\chi_{[y_i,y_{i+1}]}\end{pmatrix}\to 0,\ \text{as}\ N\to\infty,$$

here θ_i^u and θ_i^v are chosen such that $\int_{y_i}^{y_{i+1}}u(x)\mathrm{d}x=u(\theta_i^u)h$ and $\int_{y_i}^{y_{i+1}}v(x)\mathrm{d}x=v(\theta_i^v)h$ when the mean value theorem is applied, and the continuity of $u(x)$ and $v(x)$ on the internal $[0,1]$ is applied in the last step. Thus combing this result and the density of $D(A)$ in the state space H, we obtain (A_2). Moreover, let $(V^0,I^0)\in H$ be the initial value of (10) and set $(V_h(0),I_h(0))^\top=P_N(V^0,I^0)^\top$ be the initial data of (15), then (A_2) implies that $(V_h(0),I_h(0))$ convergent to (V^0,I^0) in the sense of

$$E_N(V_h(0),I_h(0))^\top\to(V^0,I^0)^\top,\ \text{as}\ N\to\infty.$$

In a word, we have completed the verification of (A_1)-(A_3) and (C_1)-(C_2) and this means that the solutions to the discrete systems (15) strongly converge to the solution of (2), i.e., as $N\to\infty$,

$$\left\|E_NT_N(t)P_N\begin{pmatrix}u(x)\\v(x)\end{pmatrix}-T(t)\begin{pmatrix}u(x)\\v(x)\end{pmatrix}\right\|_H\to 0,\ \ \forall\ \begin{pmatrix}u(x)\\v(x)\end{pmatrix}\in H. \tag{31}$$

5 Return to the original system (1)

Now, all results for the systems (3) and (15) are going to be translated into the system (1) and it's semi-discretized systems, respectively. Recall that the state space H and the system operator A corresponding to (3) have defined in Remark 1. Similarly, we introduce the state space H_O and the system operator A_O for the system (1). The space H_O is $[L^2(0,1)]^2$ with the inner product given by: for any $(p(\cdot),q(\cdot))^\top,(u(\cdot),v(\cdot))^\top\in[L^2(0,1)]^2$

$$\left\langle\begin{pmatrix}p(x)\\q(x)\end{pmatrix},\begin{pmatrix}u(x)\\v(x)\end{pmatrix}\right\rangle_{H_0}=\int_0^1\frac{p(x)\overline{u(x)}}{C(x)}+\frac{q(x)\overline{v(x)}}{L(x)}\mathrm{d}x.$$

Define the system operator A_O on H_O through

$$A_O \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \begin{pmatrix} -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{v(x)}{C(x)}\right) \\ -\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u(x)}{L(x)}\right) \end{pmatrix}, \tag{32}$$

$$D(A_0) = \left\{ \begin{pmatrix} u(\cdot) \\ v(\cdot) \end{pmatrix} \in [H^1(0,1)]^2 : u(0) = 0,\ u(1) = Rv(1) \right\}. \tag{33}$$

Thus, the system (1) can be transformed into abstract Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} Q(t,x) \\ \phi(t,x) \end{pmatrix} = A_O \begin{pmatrix} Q(t,x) \\ \phi(t,x) \end{pmatrix}, \qquad \begin{pmatrix} Q(0,x) \\ \phi(0,x) \end{pmatrix} = \begin{pmatrix} Q^0(x) \\ \phi^0(x) \end{pmatrix} \in H_O.$$

Furthermore, the relations $V = Q/C$ and $I = \phi/L$ determine a mapping $\Psi : H_O \to H$ given by

$$\begin{pmatrix} V(x) \\ I(x) \end{pmatrix} = \Psi \begin{pmatrix} Q(x) \\ \phi(x) \end{pmatrix} := \begin{pmatrix} \frac{1}{C(x)} & 0 \\ 0 & \frac{1}{L(x)} \end{pmatrix} \begin{pmatrix} Q(x) \\ \phi(x) \end{pmatrix}, \ \forall \begin{pmatrix} Q(x) \\ \phi(x) \end{pmatrix} \in H_O$$

The operators A, A_O and Ψ have following basic properties.

Proposition 2 *The operators A, A_O and Ψ satisfies:*
(1)The operator Ψ is an isometric isomorphism from H_0 to H.
(2)Let Ψ^{-1} be the inverse operator of Ψ, then the operators A and A_O are similar, i.e. $A_O = \Psi^{-1} A \Psi$.

Proof: (1) Obviously, Ψ is a linear operator from H_0 to H and invertible. The inverse of Ψ is $\Psi^{-1} = \begin{pmatrix} C(x) & 0 \\ 0 & L(x) \end{pmatrix}$ and also a linear operator from H to H_0. This implies that the operator Ψ is an isomorphism. Because the identity

$$\left\| \Psi \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_H^2 = \int_0^1 \frac{|u(x)|^2}{C(x)} + \frac{|v(x)|}{L(x)} \mathrm{d}x = \left\| \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_0}^2$$

holds, we know that Ψ is isometric.

(2) For any $\begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in D(A) = D(A_O)$, we have $\Psi^{-1} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in D(A_O)$

$$A_O \Psi^{-1} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = -\begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \Psi^{-1} A \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

which gives $A_O = \Psi^{-1} A \Psi$. $\square$

Now, we can give exponential stability result of the original system (1).

Theorem 6 *Let the assumptions H_1-H_3 hold and the semigroup on the space H_O generated by the operator A_O be $T_O(t)$, then $T_O(t)$ is exponentially stable.*

Proof: Recall that the semigroup $T(t)$ given in Remark 1 satisfies $\|T(t)\|_H \leq Me^{-\omega t}$ for some positive constants M and ω. However, it follows from $A_O = \Psi^{-1}A\Psi$ and similar semigroup theory in [36] that $T_O(t) = \Psi^{-1}T(t)\Psi$. Therefore, from (1) of Proposition 2 we have

$$\|T_O(t)\|_{H_O} = \|\Psi^{-1}T(t)\Psi\|_{H_O} = \|T(t)\|_H \leq Me^{-\omega t},$$

which means that $T_O(t)$ is exponentially stable. $\square$

Certainly, we can obtain Theorem 6 by the same method in Theorem 1. However, this indirect method for obtaining exponential stability of the original system (1) is important in studying the uniform exponential stability of it's discrete systems. From semi-discretization scheme (15) and the relations $V_j(t) = Q_j(t)/C_j$ and $I_j(t) = \phi_j(t)/L_j$, we obtain semi-discretization scheme for (1)

$$\begin{cases} C_j \frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}}\left(\frac{Q_j(t)}{C_j}\right) = -\delta_x\left(\frac{\phi_j(t)}{L_j}\right), \\ L_j \frac{\mathrm{d}}{\mathrm{d}t}\delta_{\frac{1}{2}}\left(\frac{\phi_j(t)}{L_j}\right) = -\delta_x\left(\frac{Q_j(t)}{C_j}\right), \quad j = 0, 1, \cdots N, \\ Q_0(t) = 0, \quad Q_{N+1}(t) = RC_{N+1}L_{N+1}^{-1}\phi_{N+1}(t), \\ Q(0, y_i) \approx Q^0(y_i) = Q_i^0, \ \phi(0, y_i) \approx \phi^0(y_i) = \phi_i^0, i = 0, \cdots N+1. \end{cases} \tag{34}$$

The energy of discrete system (34) is

$$E_{Oh}(t) = \frac{h}{2}\sum_{j=0}^{N}\left[C_j\left|\delta_{\frac{1}{2}}\left(\frac{Q_j(t)}{C_j}\right)\right|^2 + L_j\left|\delta_{\frac{1}{2}}\left(\frac{\phi_j(t)}{L_j}\right)\right|^2\right],$$

Now we show that the system (34) is uniform exponentially stable in the sense of Definition of 1. The state space of the discrete system (34) is $H_{ON} =: \left[\mathbb{R}^{N+1}\right]^2$ with the inner product

$$\langle(\widetilde{p}_h, \widetilde{q}_h), (\widetilde{u}_h, \widetilde{v}_h)\rangle_{ON} = h\sum_{j=0}^{N}\left[C_j\delta_{\frac{1}{2}}\frac{\widetilde{p}_j}{C_j}\delta_{\frac{1}{2}}\frac{\widetilde{u}_j}{C_j} + L_j\delta_{\frac{1}{2}}\frac{\widetilde{q}_j}{L_j}\delta_{\frac{1}{2}}\frac{\widetilde{v}_j}{L_j}\right],$$

in which $(\widetilde{p}_h, \widetilde{q}_h), (\widetilde{u}_h, \widetilde{v}_h) \in H_{ON}, \widetilde{p}_0 = \widetilde{u}_0 = 0$, $\widetilde{q}_{N+1} = k'\widetilde{p}_{N+1}$, $\widetilde{v}_{N+1} = k'\widetilde{u}_{N+1}$, and $k' = R^{-1}C_{N+1}^{-1}L_{N+1}$ are used. Let Ψ_N be matrixes of order $2N+2$ given by

$$\Psi_N = \mathrm{diag}\left\{\frac{1}{c_0}, \cdots, \frac{1}{c_N}, \frac{1}{L_0}, \cdots, \frac{1}{L_N}\right\},$$

which is well-defined since the assumption H_1 is right. Using these matrixes, we define the linear operators from H_{ON} to H_N by

$$\begin{pmatrix} p_h \\ q_h \end{pmatrix} = \Psi_N \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix}, \ \forall \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix} \in H_{ON}.$$

Thus we can get the results similar to those of Proposition 2 and Theorem 6, which are main results of this paper.

Proposition 3 *For any fixed positive integer N, the operator Ψ_N is an isometric isomorphism from H_{0N} to H_N.*

Proof: Obviously, for a fixed positive integer N, Ψ_N is an isomorphism from H_{0N} to H_N since it is invertible. The inverse of Ψ_N is $\Psi_N^{-1} = \operatorname{diag}\{c_0, \cdots, c_N, L_0, \cdots, L_N\}$. Because the identity

$$\left\| \Psi_N \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix} \right\|_{H_N}^2 = h \sum_{j=0}^{N} C_j \left| \delta_{\frac{1}{2}} \left(\frac{\widetilde{p}_j}{c_j} \right) \right|^2 + h \sum_{j=0}^{N} L_j \left| \delta_{\frac{1}{2}} \left(\frac{\widetilde{q}_j}{L_j} \right) \right|^2 = \left\| \begin{pmatrix} \widetilde{p}_h \\ \widetilde{q}_h \end{pmatrix} \right\|_{H_{0N}}^2$$

holds, we know that Ψ_N is isometric.

Theorem 7 *Define operator A_{ON} by the formulas $A_{ON} = \Psi_N^{-1} A_N \Psi$. Then the abstract Cauchy problem*

$$\frac{d}{dt} \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = A_{ON} \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} \tag{35}$$

determined by the operator A_{ON} is equivalent to the discrete system (34) without initial data. Furthermore, the discrete system (34) is uniform exponentially stable with respect to energy $E_{Oh}(t)$.

Proof: It is easy to see that the discrete system (15) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = A_N \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} \tag{36}$$

from (26) and (30). Let $Q_h(t) = (Q_1(t), \cdots, Q_{N+1}(t))$, $\phi_h(t) = (\phi_0(t), \cdots, \phi_N(t))$ and $(Q_h(t), \phi_h(t))^\top$ be state variables of the discrete system (34). Then we have the identity

$$\begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = \Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix}$$

since $V_j(t) = Q_j(t)/C_j$ and $I_j(t) = \phi_j(t)/L_j$ for $j = 0, 1, \cdots, N+1$. Substituting the identity above into (36), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = A_N \Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix},$$

which is equivalent to the discrete system (34). This means that the abstract Cauchy problem corresponding to the discrete system (34) without initial data is

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = A_{ON} \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix} = \Psi_N^{-1} A_N \Psi_N \begin{pmatrix} Q_h(t) \\ \phi_h(t) \end{pmatrix}. \tag{37}$$

That is to say $A_{ON} = \Psi_N^{-1} A_N \Psi_N$. Let $T_{ON}(t)$ and $T_N(t)$ be semigroups generated by A_{ON} and A_N respectively, we have $T_O(t) = \Psi^{-1} T(t) \Psi$. Therefore, from Proposition 3 and Theorem 15 we know

$$\|T_{ON}(t)\|_{H_{ON}} = \|\Psi_N^{-1} T_N(t) \Psi\|_{H_{ON}} = \|T_N(t)\|_{H_N} \leq M e^{-\omega t},$$

which means that $T_{ON}(t)$ is uniform exponentially stable, i.e. the the exponential decay rates of $T_{ON}(t)$ are independent of N. But if you restate this result in the language of energy, you can obtain that the discrete systems (34) are uniform exponentially stable with respect to energy $E_{Oh}(t)$. □

Finally, the solution to the discrete system (34) is convergent to the one of the continuous system (1) in the meaning of Trotter-Kato Theorem. In fact, constructing the extensions E_{ON} from H_{ON} to H_O by $E_{ON} = \Psi^{-1} E_N \Psi_N$ and the projecting operators P_{ON} from H_O to H_{ON} by $P_{ON} = \Psi_N^{-1} E_N \Psi$, we have the following convergence result.

Theorem 8 *As $N \to \infty$, we have*

$$\left\| E_{ON} T_{ON}(t) P_{ON} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} - T_O(t) \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_O} \to 0, \quad \forall \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \in H_O. \tag{38}$$

Proof: From the relation $T_O(t) = \Psi^{-1} T(t) \Psi$ of the semigroups, Proposition 2 and Proposition 3 we have

$$\begin{aligned} & \left\| [E_{ON} T_{ON}(t) P_{ON} - T_O(t)] \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_O} \\ = & \left\| \Psi^{-1} [E_N T_N(t) P_N - T(t)] \Psi \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H_O} \\ = & \left\| [E_N T_N(t) P_N - T(t)] \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \right\|_{H}. \end{aligned}$$

The identities above and (31) imply that (38) holds and the proof of the theorem is finished. □

6 Numerical simulations

In this section, we show the effectiveness of our numerical approximating schemes (15) or (36) and (37) through some numerical experiments. Recall that $\Psi_N = \operatorname{diag}\left\{\frac{1}{c_0}, \cdots, \frac{1}{c_N}, \frac{1}{L_0}, \cdots, \frac{1}{L_N}\right\}$ and $A_{ON} = \Psi_N^{-1} A_N \Psi_N$ are given in last section. Because the operator A_N and A_{ON} are similar, we only give eigenvalue distributions of A_N to analyze uniformly exponential stability of (15) or (36). For this purpose, we should express the operator A_N as matrix. Let $G_h = \operatorname{diag}\{0, \cdots, 0, 1\}$,

$$B_h = \begin{pmatrix} 1 & 1 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \\ & & & 1 \end{pmatrix}, \quad M_h = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix}$$

belong to $\mathbb{C}^{(N+1)\times(N+1)}$. Set

$$\Phi_N = \frac{1}{2}\begin{pmatrix} B_h^\top & 0 \\ kG_h & B_h \end{pmatrix}, \text{ and } \Omega_N = \frac{1}{h}\begin{pmatrix} -kG_h & M_h^\top \\ -M_h & 0 \end{pmatrix}.$$

Then (36) is equivalent to

$$\Phi_N \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = \Psi_N \Omega_N \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix}, \tag{39}$$

in which $A_N = \Phi_N^{-1}\Psi_N\Omega_N$ is used. It is easy to see that Φ_N is corresponding to the average operator of time derivative of (15). If one replace B_h by identity operator, then the classical finite difference scheme of (3) is easily restored from (39), i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix} = \Psi_N \Omega_N \begin{pmatrix} V_h(t) \\ I_h(t) \end{pmatrix}, \tag{40}$$

If $C(x) = L(x) = 1$, the numerical approximating scheme (15) degenerates to the numerical approximating scheme (2.5) of [25]. The authors pointed out that the numerical approximating scheme (2.5) in [25, Lemma 5.1] is essentially similar to the mixed finite element scheme of [12, 18]. However, this is not accidental, mainly due to the intervention of the average operator.

Now we explain the significance of the discrete scheme (39) or (15). We plot two figures in Figure 1 and Figure 2, respectively. Figure 1 depicts the maximal real parts of A_N and $\Psi_N\Omega_N$ for $N = 40 : 5 : 400$. Figure 2 depicts the distributions of the eigenvalues of A_N and $\Psi_N\Omega_N$ with $N = 500$.

We see that the real parts of the eigenvalues of $\Psi_N\Omega_N$ approach to zero and those of A_N approach to a negative number from both figures. In both figures, we take $k = R = 1$, $C(x) = \ln(1 + x)$ and $L(x) = e^x$. Numerical simulation results show that the classical finite difference scheme of (40) is not uniformly

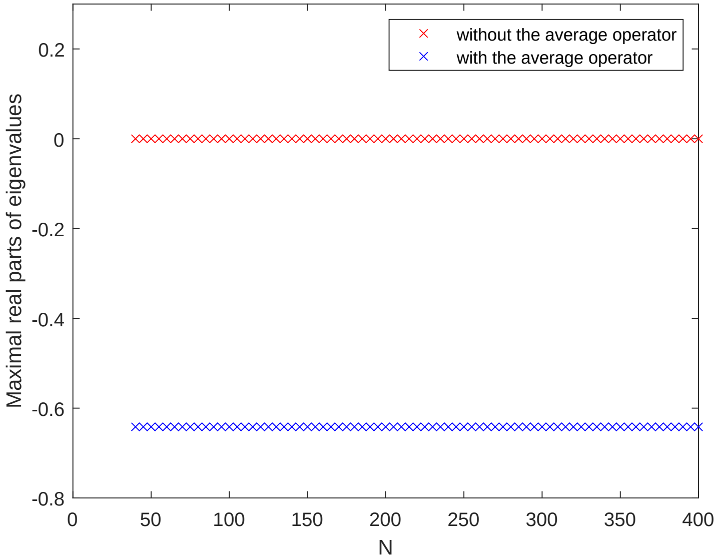

Fig. 1 Maximal real parts of eigenvalues of the semi-discrete schemes

exponentially stable. This conclusion is consistent with earlier research results of [21]. However, Figure 1 manifests (15) is perhaps uniformly exponentially stable and this is in accordance with theory results of section 3.

7 Concluding remarks

Transmission line is a basic structure of circuit and play important role in physics and engineering. This paper is devoted to uniformly exponentially stable approximations for the transmission line with varying capacity and inductance. This means that we study it from the viewpoints of the numerical approximating and control theory. It is well-known that there are many discretization methods to discretize the spatial variables. It is nontrivial to pick one which preserves exponential stability among so many semi-discretization methods. On the other hand, if the capacity parameter and the inductance parameter are constant, many existing results can be applied and there is no any challenge. To bypass the troubles brought by variable coefficients, suitable similar transforms are introduced. Uniform exponential stability and uniform observability are then smoothly obtained based on the method for the wave equations. These results have potential applications in uniform controllability, the approximation of the control problem and state reconstruction etc. They deserve to be investigated at length in the further research.

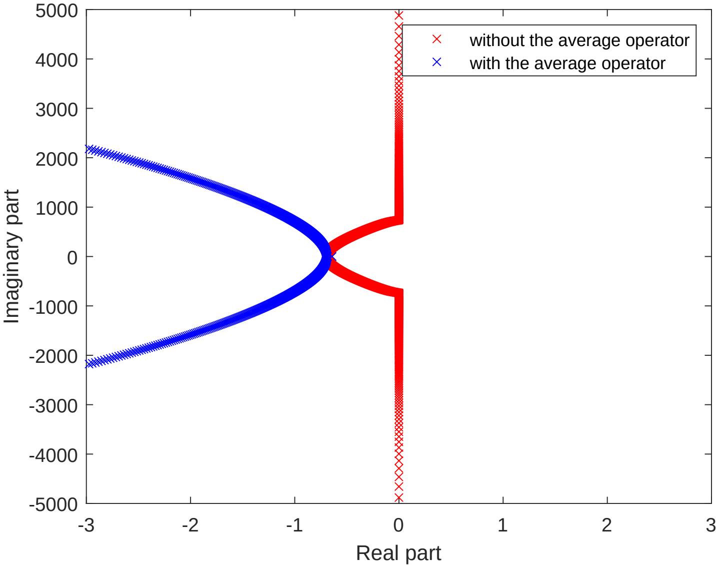

Fig. 2 Maximal real parts of eigenvalues of the semi-discrete scheme with $N = 500$.

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